$$F(x) = \begin{cases} \frac{x^2}{4} & , 0 < x < 1 \\ \frac{x}{2} - \frac{1}{4} & , 1 < x < 2 \\ \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4} & , 2 < x < 3 \\ 1 & , \text{otherwise} \end{cases}$$

5. $f(x) = \begin{cases} kx^2 e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$ r^{th} moment

To find K:-

$2x = dt$

WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} (Kx^2 e^{-x})\,dx = 0 \Rightarrow K\int_0^{\infty} x^2 e^{-x}\,dx$$

$u = x^2 \quad v = e^{-x}$

$u' = 2x \quad v_1 = -e^{-x}$

$u'' = 2 \quad v_2 = e^{-x}$

$u''' = 0 \quad v_3 = -e^{-x}$

$$K\left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$K[0 - (0 - 0 - 2e^0)] = 1$$

$$K[2] = 1$$

$$K = \frac{1}{2}$$

To find MGF:-

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \frac{1}{3} e^{-x/3}\,dx = \frac{1}{3}\int_0^{\infty} e^{(t-1/3)x}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-(1/3-t)x}}{-(1/3-t)}\right]_0^{\infty}$$

$$= \frac{1}{3} \times \frac{1}{-(\frac{1}{3}-t)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{1}{-\frac{3}{3}+3t}[0-1] = \frac{-1}{3t-1} = \frac{1}{1-3t}$$

$$M_x(t) = \frac{1}{1-3t}$$

To find mean :-

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\cancel{= \int_0^{\infty} x \frac{1}{3} e^{-x/3}\,dx = \frac{1}{3}\int_0^{\infty} x e^{-x/3}\,dx}$$

$$E(X) = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{1}{1-3t}\right)\right]_{t=0}$$

$$= \left[\frac{1}{(1-3t)^2} \times -1 \times -3\right]_{t=0} = \left[\frac{3}{(1-3t)^2}\right]_{t=0}$$

$$E(X) = 3$$

To find the first 4 moments:

$$\mu_r' = \frac{3\,2^{r+1}}{(r+2)(r+3)}$$

Put $r=1$

$$\mu_1' = \frac{3\,2^{1+1}}{(1+2)(1+3)} = \frac{3\,2^2}{3 \cdot 4} = 1$$

Put $r=2$

$$\mu_2' = \frac{3\,2^3}{(4)(5)} = \frac{6}{5}$$

Put $r=3$

$$\mu_3' = \frac{\cancel{3}\,2^{\cancel{4}\,3}}{(5)(\cancel{6})\,2} = \frac{8}{5}$$

Put $r=4$

$$\mu_4' = \frac{\cancel{3}\,2^{\cancel{5}\,4}}{(\cancel{6})(7)} \;\; {\scriptstyle 2} = \frac{16}{7}$$

2. Let X be a RV with PDF $f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x>0 \\ 0, & \text{otherw} \end{cases}$

Find $P(X>3)$, MGF, mean, variance

$$P(X>3) = \int_3^{\infty} f(x)\,dx$$

$$= \int_3^{\infty} \frac{1}{3} e^{-x/3}\,dx = \frac{1}{3}\int_3^{\infty} e^{-x/3}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty} = \frac{1}{3}\left[\frac{e^{-\infty}}{-1/3} - \frac{e^{-3/3}}{-1/3}\right]$$

$$= \frac{1}{\cancel{3}} \times -\cancel{3}\,[0 - e^{-1}]$$

$$P(X>3) = e^{-1}$$

Tutorial - 2

1. The pdf of a x is $f(x) = \begin{cases} Kx(2-x), & 0<x<2 \\ 0 & ,\text{otherwise} \end{cases}$

find the r^{th} moment about origin and hence find the first 4 moment

Given,

$$f(x) = \begin{cases} Kx(2-x), & 0<x<2 \\ 0 & ,\text{otherwise} \end{cases}$$

To find K :-

WKT, $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

then here,

$$\int_0^2 Kx(2-x) = 1 \Rightarrow \int_0^2 K(2x-x^2) = 1$$

$$K\int_0^2 2x-x^2 = 1$$

$$K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[\frac{\cancel{2}2^2}{\cancel{2}} - \frac{2^3}{3}\right] = 1$$

$$K\left[4 - \frac{8}{3}\right] = 1 \Rightarrow K\left[\frac{4}{3}\right] = 1$$

$$\boxed{K = \frac{3}{4}}$$

$$f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

To find r^{th} moment :-

$$E(x^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$$

$$= \int_0^{\infty} x^r \frac{1}{2} x^2 e^{-x})\,dx$$

$$= \frac{1}{2} \int_0^{\infty} x^{r+2} e^{-x})\,dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x} x^{(r+3)-1}\,dx$$

$$\mu_r' = \frac{1}{2} \Gamma(r+3)$$